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MIN–MAX RELATIONS FOR DIRECTED GRAPHS*

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We prove the following. Let $D = (V, A)$ and $D' = (V, A')$ be directed graphs, both with vertex set V, where D' is acyclic such that each pair of source and sink of D' is connected by a directed path in D'. Suppose that each nonempty proper subset of V which is not entered by any arrow of D', is entered by at least k arrows of D. Then A can be split into classes $A_1, \ldots, A_k$ such that the directed graph $(V, A' \cup A_i)$ is strongly connected, for each i.

This theorem contains as special cases Menger's theorem, Gupta's theorem, Edmonds' branching theorem, a 'bi-branching theorem', a special case of a conjecture of Edmonds and Giles, and a theorem of Frank. The proof yields a polynomial algorithm for finding the splitting as required.

Besides, a slight extension of the Lucchesi–Younger theorem is given.

0. Introduction

Let $D = (V, A)$ and $D' = (V, A')$ be directed graphs, both with vertex set V. Call a subset A'' of A a *strong connector* (*for* D') if the directed graph $(V, A' \cup A'')$ is strongly connected. If V' is a nonempty proper subset of V such that no arrow of D' enters V', the set of arrows of D entering V' is called a *strong cut* (*induced by* D').

We prove the following theorem.

> *If D' is acyclic and each pair of source and sink of D' is connected by a directed path in D', then the maximum number of pairwise disjoint strong connectors for D' is equal to the minimum size of a strong cut induced by D'.* (0.1)

This min–max relation has the following corollaries.

(i) *Menger's theorem* [19]. Let r and s be two vertices of the directed graph $D = (V, A)$. If no set with less than k arrows intersects each directed path from r to s, then there are k pairwise arrow-disjoint such paths. This follows from (0.1) by taking $A' = \{(v, w) \mid v = s \text{ or } w = r\}$. A subset A'' of A is a strong connector for $D' = (V, A')$ if and only if A'' contains a path from r to s.

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(ii) *Gupta's theorem* [11]. Let $G = (V, E)$ be a bipartite graph of minimum degree k. Then E contains k pairwise disjoint subsets, each covering V. For, if V' and V'' are the two colour classes of G, let D arise from G by orienting all edges of G from V'' to V', and let $A' = \{(v', v'') \mid v' \in V', v'' \in V''\}$. Now a subset A'' of A is a strong connector for D' if and only if A'' covers V.

(iii) *Edmonds' branching theorem* [2]. Let $D = (V, A)$ be a directed graph, and let r be a vertex of D. If each nonempty subset of $V \backslash \{r\}$ is entered by at least k arrows of D, then A contains k pairwise disjoint r-branchings. Here an *r-branching* is a set A'' of arrows such that each vertex of D is reachable by a directed path from r in A''. The result follows from (0.1) by taking $A' = \{(v, r) \mid v \in V \backslash \{r\}\}$. Then A'' is an r-branching if and only if A'' is a strong connector for D'.

(iv) *A bi-branching theorem.* Let $D' = (V, A)$ be a directed graph, and let V be split into classes V' and V''. Suppose each nonempty subset of V' is entered by at least k arrows, and each nonempty subset of V'' is left by at least k arrows. Then A contains k pairwise disjoint bi-branchings. Here a subset A'' of A is called a *bi-branching* (with respect to the splitting V', V'') if each vertex in V' is the end point of some directed path in A'' starting in V'', and each vertex in V'' is the starting point of some directed path in A'' ending in V'. So for $V'' = \{r\}$ we obtain r-branchings. The result follows from (0.1) by taking $A' = \{(v', v'') \mid v' \in V', v'' \in V''\}$. Then A'' is a bi-branching if and only if A'' is a strong connector for D'.

(v) *A special case of a conjecture of Edmonds and Giles* [3]. Let $D' = (V, A')$ be a directed graph, and let C be a subset of A' such that each directed cut of D' contains at least k arrows of C. (A *directed cut* is the set of arrows entering some nonempty proper subset V' of V, provided that no arrow leaves V'.) Edmonds and Giles conjectured that C can be split into k classes $C_1, \ldots, C_k$ such that each C_i intersects each directed cut (i.e., such that contracting the arrows in C_i makes D' strongly connected). Although the general conjecture appeared to be not true (cf. [20]), in the special case that D' is acyclic and each pair of source and sink of D' is connected by a directed path, the conjecture follows from (0.1) by taking A to be the collection of arrows in C with reversed orientation. Then a subset of A is a strong connector for D' if and only if the corresponding subset of C intersects each directed cut of D'. (This special case of the conjecture was announced independently by D.H. Younger.)

(vi) *A theorem of Frank* [5]. Let $D = (V, A)$ be a directed graph, let r be a vertex of D, and let $\mathcal{F}$ be a collection of subsets of $V \backslash \{r\}$ closed under taking unions and intersections. Suppose that each nonempty set in $\mathcal{F}$ is entered by at least k arrows in D. Then A can be split into classes $A_1, \ldots, A_k$ such that each nonempty set in $\mathcal{F}$ is entered by at least one arrow in each of the A_i. This follows from (0.1) by taking A' to be the set of all pairs (v, w) which do not enter any set in $\mathcal{F}$. (Possibly D' is made acyclic by contracting strong components.) Actually, Frank proved the more general result where it suffices to require $\mathcal{F}$ to be closed under taking unions and intersections of *intersecting* sets in $\mathcal{F}$.

Remark. The condition of D' being acyclic is not essential. Requiring D' to satisfy the conditions after contracting its strong components is sufficient. Actually, it is not difficult to see that (0.1) is equivalent to: let $D = (V, A)$ be a directed graph, and let $\mathcal{F}$ be a collection of subsets of V closed under taking unions and intersections, such that no V_1, V_2, V_3 in $\mathcal{F}\backslash\{\emptyset, V\}$ have $V_1 \cap V_2 \cap V_3 = \emptyset$ and $V_1 \cup V_2 \cup V_3 = V$. If each set in $\mathcal{F}\backslash\{\emptyset, V\}$ is entered by at least k arrows of D, then A contains k pairwise disjoint sets $A_1, \ldots, A_k$ such that each set in $\mathcal{F}\backslash\{\emptyset, V\}$ is entered by an arrow in each A_i.

The corollaries (i)–(vi) are not independent; one easily derives the following implications: (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i), (vi) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (ii), and (v) $\Rightarrow$ (ii). In fact, our proof essentially shows some more implications.

In Section 1 we first give, for the sake of completeness, a proof of Edmonds' branching theorem (iii), by slightly adapting the proof of Lovász [16]. Second, in Section 2, we prove the following general theorem on pairs of submodular functions. (A function f defined on the subsets of a set X is called *submodular* if $f(X') + f(X'') \geqslant f(X' \cap X'') + f(X' \cup X'')$ for all subsets X' and X'' of X.)

> *Let f_1 and f_2 be integral submodular set-functions on a set X, such that $f_i(X') \geqslant \max\{|X'|, k\}$ for each nonempty subset X' of X, and $i = 1, 2$. Then X can be split into classes $X_1, \ldots, X_k$ such that $f_i(X') \geqslant \Sigma_{j=1}^{k} \max\{|X' \cap X_j|, 1\}$ for each nonempty subset X' of X and $i = 1, 2$.* (0.2)

Actually, this is a theorem on the splitting of vectors in polymatroids (cf. [1] and the remark in Section 2). It generalizes the edge-colouring theorems of König [13] and Gupta [11] in a similar way as Edmonds' matroid intersection theorem [1] generalizes the König–Egerváry theorem [4, 14] on matchings in bipartite graphs.

Third, in Section 3, we show that (0.2) allows us to glue branchings together to form bi-branchings, and thus to extend (iii) to (iv). In Section 4 we deduce, with some induction arguments, (v) from (iv). Finally, in Section 5, we apply a direct construction to obtain the general Theorem (0.1) from (v). Note that, by replacing arrows by parallel arrows, one easily obtains a 'weighted' version of (0.1).

In Section 6 we use this last 'direct construction' also to observe that the following can be derived from the Lucchesi–Younger theorem [18].

Let $D = (V, A)$ and $D' = (V, A')$ be directed graphs, such that for any arrow (v, w) of D there are vertices v' and w' in V, and directed paths in D' from v to v', from w' to v', and from w' to w (cf. Fig. 1, where the wriggled lines stand for directed paths in D'). Let $l : A \to \mathbb{Z}_+$ be some 'length' function. Then the minimum length of a strong connector for D' is equal to the maximum number of strong cuts induced by D' such that no arrow a is in more than $l(a)$ of these strong cuts. (0.3)

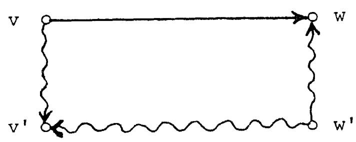

Fig. 1.

If A is the collection of reversed arrows of D', the assumption is obviously satisfied and assertion (0.3) is just the Lucchesi–Younger theorem. If D' is as in (i), (ii), (iii) and (iv) above, we obtain, successively, an (easy) theorem of Fulkerson [7], König's theorem on minimum coverings in a bipartite graph [15], Fulkerson's branching theorem [9], and another 'bi-branching theorem': if the vertex set V of the directed graph $D = (V, A)$ is split into classes V' and V'', and if $c : A \to \mathbb{Z}_+$ is some capacity function, then the minimum capacity of a bi-branching is equal to the maximum number of nonempty proper subsets $V_1, \ldots, V_k$ of V such that $V_i \subset V'$ or $V' \subset V_i$ for each i, and no arrow a of D enters more than $c(a)$ of the V_i.

The conditions for D and D' given in (0.3) are less restrictive than those given in (0.1). In fact, for acyclic D', there is a directed path between each pair of source and sink, if and only if *each* pair (v, w) of vertices of D' is connected by a path of the form of the wriggled lines in Fig. 1. In (0.1) we may not relax the conditions on D' to those given in (0.3), as is shown by the counterexample to the conjecture of Edmonds and Giles (cf. (iv) above). Moreover, if $D = (V, A)$ and $D' = (V, A')$ are as in Fig. 2, where light and heavy lines represent the arrows of D and D', respectively, then any strong connector for D' has cardinality at least 3, whereas any strong cut induced by D' contains at least 2

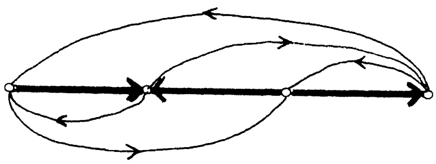

Fig. 2.

arrows of D. Since $|A|=5$, it is not sufficient to require in (0.1) or (0.3) D' to be weakly connected.

In Section 7 we discuss some generalizations of the results, in terms of sub- and supermodular functions defined on directed graphs, following the lines set out by Edmonds and Giles [3] and Frank [5]. In fact, we give a generalization of (0.3) which slightly extends the theorem of Frank. We also comment on similar extensions of (0.1) and (0.2).

Finally, in Section 8, we formulate the results in terms of polyhedra and linear programming, and this yields, by the ellipsoid method as described in [10], the polynomial solvability of most of the problems. Besides, our proof of (0.1) and (0.2) above will be polynomially constructive (using the fact that the minimum value of a submodular set-function can be found in polynomial time [10]), yielding a polynomial algorithm for optimum packing of strong connectors.

Some terminology. Above we gave already the, rather standard, definitions of submodular function, r-branching and directed cut, and we introduced the notion of bi-branching. A function g is *supermodular* if $-g$ is submodular. We shall sometimes use the easy observation that if f is a submodular, and g is a supermodular set-function on X with $g(X') \leq f(X')$ for all $X' \subset X$, then the collection of sets X' with $g(X') = f(X')$ is closed under taking unions and intersections.

The *indegree* (*outdegree*, respectively) of a set V' of vertices of a directed graph $D=(V, A)$ is the number of arrows of D entering V' (leaving V', respectively), and is denoted by $d_A^-(V')$ ($d_A^+(V')$, respectively).

If c is a rational-valued function defined on a set X, and X' is a subset of X, then, by definition,

$$c(X') := \sum_{x \in X'} c(x) .$$

If c is called a capacity function, then $c(X')$ is the *capacity* of X'.

We note that directed graphs may have multiple arrows, but that we often speak of 'the arrow (v, w)', where 'an arrow from v to w' would be formally more correct.

1. Edmonds' branching theorem

We first give, for the sake of completeness, a proof of a theorem of Edmonds [2], by adapting the method of Lovász [16]. By Edmonds' branching theorem usually is understood the case where $V_1 = \cdots = V_k = \{r\}$.

Theorem 1. *Let $D = (V, A)$ be a directed graph, and let $V_1, \dots, V_k$ be subsets of V. Suppose $d_A^-(V') \geq h(V')$ for each nonempty subset V' of V, where $h(V')$ denotes the number of i with $V' \cap V_i = \emptyset$. Then A can be split into classes $A_1, \dots, A_k$ such that for each i and each v in V there is a directed path in A_i starting in V_i and ending in v.*

Proof. By induction on $\Sigma_{i=1}^k |V \setminus V_i|$, the case $V_1 = \cdots = V_k = V$ being trivial. Denote by

$$h_{X_1, \dots, X_t}(V') \tag{1.1}$$

the number of $i = 1, \dots, t$ with $V' \cap X_i = \emptyset$.

Suppose that $V_1 \neq V$ (say), and consider the collection $\mathscr{F}$ of subsets V' of V with

$$d_A^-(V') = h_{V_2, \dots, V_k}(V'). \tag{1.2}$$

Note that here the inequality $\geq$ always holds, and that (1.2) implies that $V' \cap V_1 \neq \emptyset$. Since the left-hand side of (1.2) is a submodular, and the right-hand side is a supermodular function, the collection $\mathscr{F}$ is closed under unions and intersections. Moreover, $V \in \mathscr{F}$, so there exists a minimal set V' in $\mathscr{F}$ with $V' \not\subseteq V_1$. As

$$d_A^-(V' \setminus V_1) \geq h_{V_1, \dots, V_k}(V' \setminus V_1) > h_{V_2, \dots, V_k}(V') = d_A^-(V'), \tag{1.3}$$

there is an arrow $a = (v, w)$ from $V' \cap V_1$ to $V' \setminus V_1$. We show that

$$d_{A \setminus a}^-(V'') \geq h_{V_1 \cup w, V_2, \dots, V_k}(V''), \tag{1.4}$$

for each nonempty subset V'' of V. By induction this implies the theorem, as we can split $A \setminus \{a\}$ into classes as required with respect to $V_1 \cup \{w\}, V_2, \dots, V_k$, and hence, by adding the arrow a to the first class, we obtain a splitting of A as required for $V_1, \dots, V_k$.

To show (1.4), suppose $V'' \neq \emptyset$ violates (1.4). Since

$$d_A^-(V'') \geq h_{V_1, \dots, V_k}(V'') \geq h_{V_1 \cup w, V_2, \dots, V_k}(V'') > d_{A \setminus a}^-(V'') \geq d_A^-(V'') - 1, \tag{1.5}$$

we know that a enters V'', that $w \in V''$, and that

$$d_A^-(V'') = h_{V_1 \cup w, V_2, \dots, V_k}(V'') = h_{V_2, \dots, V_k}(V''). \tag{1.6}$$

So V'' is in $\mathscr{F}$, and hence $V' \cap V''$ is in $\mathscr{F}$. Since $V' \cap V'' \not\subset V_1$ as $w \in V' \cap V''$, and since $V' \cap V'' \neq V'$ as $v \notin V''$, this contradicts the minimality of V'. $\square$

2. Pairs of submodular functions

In order to glue branchings together to obtain bi-branchings, we prove a theorem on submodular functions, which has as direct corollaries the theorems of König [13] and Gupta [11] on edge-colourings of bipartite graphs. Also the more general theorem of De Werra [21] may be derived: if (V, E) is a bipartite graph and k is a natural number, then E can be split into classes $E_1, \ldots, E_k$ such that each vertex v is covered by $\min\{d(v), k\}$ of the E_i, where $d(v)$ denotes the degree of v.

Theorem 2. *Let f_1 and f_2 be integral submodular set-functions on a set X, such that*

$$f_i(X') \geq \max\{|X'|, k\} \tag{2.1}$$

for each nonempty subset X' of X, and $i = 1, 2$. Then X can be partitioned into classes $X_1, \ldots, X_k$ such that

$$f_i(X') \geq \sum_{j=1}^{k} \max\{|X_j \cap X'|, 1\} \tag{2.2}$$

for each nonempty subset X' of X, and $i = 1, 2$.

Proof. (i) We first prove the theorem for $k = 2$. Let $Y_1, \ldots, Y_s$ be the minimal nonempty subsets of X with $f_1(Y_j) = |Y_j|$. So the $Y_1, \ldots, Y_s$ are pairwise disjoint, since the collection of sets X' with $f_1(X') = |X'|$ is closed under taking unions and intersections. Moreover, $|Y_j| \geq 2$ for each j, since $f_1(X') \geq 2$ for all nonempty sets X'.

Similarly, let $Z_1, \ldots, Z_t$ be the minimal nonempty subsets of X with $f_2(Z_j) = |Z_j|$. Again, $Z_1, \ldots, Z_t$ are pairwise disjoint and contain at least two elements.

Hence X can be partitioned into classes X_1, X_2 such that both X_1 and X_2 intersect each of $Y_1, \ldots, Y_s, Z_1, \ldots, Z_t$. We prove that (2.2) is satisfied for this choice of X_1 and X_2. Let X' be a nonempty subset of X. If $X_1 \cap X' \neq \emptyset \neq X_2 \cap X'$, then (2.2) follows from (2.1). So we may suppose that $X_2 \cap X' = \emptyset$. Then X' does not contain any of the $Y_1, \ldots, Y_s, Z_1, \ldots, Z_t$, implying that $f_1(X') > |X'|$ and $f_2(X') > |X'|$, which proves (2.2).

(ii) In order to prove the theorem for arbitrary $k \geq 2$, let $X_1, \ldots, X_k$ be pairwise disjoint subsets of X such that (2.2) holds and such that $|X_1 \cup \cdots \cup X_k|$ is as large as possible. If $X_1 \cup \cdots \cup X_k = X$ we are finished, so suppose that $x \in X \setminus (X_1 \cup \cdots \cup X_k)$. Consider the collection $\mathcal{F}$ of all subsets X' of X with $x \in X'$ and

$$f_1(X') = \sum_{j=1}^{k} \max\{|X_j \cap X'|, 1\}. \tag{2.3}$$

Suppose $\mathcal{F} \neq \emptyset$. Since $\mathcal{F}$ is closed under unions and intersections (as the left- and right-hand sides of (2.3) are submodular and supermodular, respectively), there is a unique maximal element Y in $\mathcal{F}$. If Y intersects each of the X_i then

$$f_1(Y) = \sum_{j=1}^{k} |X_j \cap Y| = |Y \cap (X_1 \cup \cdots \cup X_k)| \leq |Y \setminus \{x\}| < |Y|, \tag{2.4}$$

contradicting (2.1). So without loss of generality we may assume that $Y \cap X_1 = \emptyset$. This implies that,

$$\text{if } x \in X' \text{ and } X' \cap X_1 \neq \emptyset, \text{ then } f_1(X') > \sum_{j=1}^{k} \max\{|X_j \cap X'|, 1\}. \tag{2.5}$$

Obviously, this is also true if $\mathcal{F} = \emptyset$.

Similarly, there exists an index j such that

$$\text{if } x \in X' \text{ and } X' \cap X_j \neq \emptyset, \text{ then } f_2(X') > \sum_{j=1}^{k} \max\{|X_j \cap X'|, 1\}. \tag{2.6}$$

If $j = 1$ one easily checks that replacing X_1 by $X_1 \cup \{x\}$ does not violate (2.2), contradicting the maximality of $X_1 \cup \cdots \cup X_k$. So suppose $j \neq 1$, say $j = 2$.

Now (2.5) and (2.6) imply

$$f_i(X') \geq \max\{|(X_1 \cup X_2 \cup \{x\}) \cap X'|, 2\} + \sum_{j=3}^{k} \max\{|X_j \cap X'|, 1\} \tag{2.7}$$

for each nonempty subset X' of X, and $i = 1, 2$. Define

$$f_i'(X') = \min_{X'' \subset X \setminus (X_1 \cup X_2 \cup \{x\})} f_i(X' \cup X'') - \sum_{j=3}^{k} \max\{|X_j \cap X''|, 1\} \tag{2.8}$$

for subsets X' of $X_1 \cup X_2 \cup \{x\}$, and $i = 1, 2$. The functions f_1' and f_2' are submodular again, and from (2.7) we know that

$$f_i'(X') \geq \max\{|X'|, 2\} \tag{2.9}$$

for each nonempty subset X' of X, and $i = 1, 2$. Hence, by part (i) above, we can split $X_1 \cup X_2 \cup \{x\}$ into classes X_1' and X_2' such that

$$f_i'(X') \geq \sum_{j=1}^{2} \max\{|X_j' \cap X'|, 1\} \tag{2.10}$$

for each nonempty subset X' of $X_1 \cup X_2 \cup \{x\}$, and $i = 1, 2$. Hence, by definition (2.8) of the f_i', the sets $X_1', X_2', X_3' = X_3, \ldots, X_k' = X_k$ form a collection of pairwise disjoint sets satisfying

$$f_i(X') \geq \sum_{j=1}^{k} \max\{|X_j' \cap X'|, 1\} \tag{2.11}$$

for each nonempty subset X' of X, contradicting the maximality of $X_1 \cup \cdots \cup X_k$. □

In fact, Theorem 2 may be considered as a theorem on the splitting of vectors in polymatroids (cf. [1]), since it can be extended easily to: *let f_1 and f_2 be integral submodular set-functions on a set X, and let $b : X \to \mathbb{Z}_+$ be such that $f_i(X') \geq \max\{b(X'), k\}$ for each nonempty subset X' of X, and $i = 1, 2$. Then there exist $b_1, \ldots, b_k : X \to \mathbb{Z}_+$ such that $b = b_1 + \cdots + b_k$ and $f_i(X') \geq \sum_{j=1}^{k} \max\{b_j(X'), 1\}$ for each nonempty subset X' of X, and $i = 1, 2$.*

3. A bi-branching theorem

Combination of Theorem 1 and Theorem 2 gives a theorem on bi-branchings.

Theorem 3. *Let $D = (V, A)$ be a directed graph, and let V be split into classes V_1 and V_2, such that any nonempty subset of V_1 (of V_2, respectively) is entered (left, respectively) by at least k arrows of D. Then A can be split into k bi-branchings.*

Proof. Let X be the set of arrows from V_2 to V_1, and define the set-functions f_1 and f_2 on X by

$$f_1(X') = \min\{d_A^-(V_1') \mid V_1' \subset V_1, \text{ and each arrow in } X' \text{ ends in } V_1'\},$$
$$f_2(X') = \min\{d_A^+(V_2') \mid V_2' \subset V_2, \text{ and each arrow in } X' \text{ starts in } V_2'\}, \tag{3.1}$$

for $X' \subset X$. It is easy to check that f_i is submodular, and that $f_i(X') \geqslant \max\{|X'|, k\}$, for each nonempty subset X' of X, and $i = 1, 2$. Hence, by Theorem 2, we can split X into classes $X_1, \ldots, X_k$ such that

$$f_i(X') \geqslant \sum_{j=1}^{k} \max\{|X_j \cap X'|, 1\} \tag{3.2}$$

for each nonempty subset X' of X, and $i = 1, 2$. Let Y_j (Z_j, respectively) be the set of heads (tails, respectively) of arrows occurring in X_j. Consider any nonempty subset V_1' of V_1, and let X_0 be the set of arrows in X with head in V_1'. If $X_0 \neq \emptyset$, by (3.2)

$$f_1(X_0) \geqslant \sum_{j=1}^{k} \max\{|X_j \cap X_0|, 1\}. \tag{3.3}$$

In particular,

$$d_A^-(V_1') \geqslant f_1(X_0) \geqslant |X_0| + |\{j | X_j \cap X_0 = \emptyset\}| = |X_0| + h(V_1'), \tag{3.4}$$

where $h(V_1')$ is the number of j with $Y_j \cap V_1' = \emptyset$. Hence, as $d_X^-(V_1') = |X_0|$, it follows that

$$d_{A'}^-(V_1') \geqslant h(V_1'), \tag{3.5}$$

where A' is the set of arrows contained in V_1. As (3.5) is true also if $X_0 = \emptyset$, (3.5) is true for each nonempty subset V_1' of V_1, and hence, by Theorem 1 we can split A' into classes $A_1', \ldots, A_k'$ such that if V_1' is a nonempty subset of $V_1 \setminus Y_j$, then at least one arrow in A_j' enters V_1', for $j = 1, \ldots, k$.

Similarly, one can split the arrows contained in V_2 into k classes $A_1'', \ldots, A_k''$ such that if V_2' is a nonempty subset of $V_2 \setminus Z_j$ then at least one arrow in A_j'' leaves V_2', for $j = 1, \ldots, k$.

It follows that $A_1' \cup X_1 \cup A_1'', \ldots, A_k' \cup X_k \cup A_k''$ yields a splitting as required. □

4. A special case of a conjecture of Edmonds and Giles

Theorem 3 is used to show the following theorem, which proves a special case of a conjecture of Edmonds and Giles [3], announced independently by D.H. Younger.

Theorem 4. *Let $D = (V, A)$ be an acyclic directed graph, such that any pair of source and sink is connected by a directed path, and let C be a subset of A such that each directed cut of D intersects C in at least k arrows. Then C can be split into classes $C_1, \ldots, C_k$ such that each class C_i intersects each directed cut.*

Proof. We prove the theorem by induction on $|V| + |C|$. Suppose the assertion does not hold for D and C, and suppose this counterexample has been chosen with $|V| + |C|$ as small as possible.

Call a subset V' of V a *kernel* for D if $\emptyset \neq V' \neq V$ and $d_A^+(V') = 0$. We may assume without loss of generality that if there is a directed path in D from v to w, then $(v, w) \in A$, as the adding of such arrows does not change the collection of kernels. So we may think of D as just a partially ordered set.

(i) *If V' is a kernel with $d_C^-(V') = k$, then $|V'| = 1$ or $|V \backslash V'| = 1$, i.e., directed cuts intersecting C in exactly k arrows are determined by sources and sinks.* For suppose V' is a kernel with $d_C^-(V') = k$ and $|V'| \geqslant 2$ and $|V \backslash V'| \geqslant 2$. Let C' be the set of arrows in C with head in V', and let C'' be the set of arrows in C with tail in $V \backslash V'$. Contracting $V \backslash V'$ to one point yields a smaller directed graph $D' = (V', A')$, with $C' \subset A'$ and each directed cut of D' intersecting C' in at least k arrows. Hence, by induction, C' can be split into classes $C'_1, \ldots, C'_k$ such that each directed cut of D' intersects each C'_i. Similarly, by contracting V', thus obtaining the directed graph D'', the projection C'' of C can be split into classes $C''_1, \ldots, C''_k$ such that each directed cut of D'' intersects each C''_i. So each C'_i and each C''_i contain exactly one of the arrows in C from $V \backslash V'$ to V', and we may assume that $C'_i \cap C''_j \neq \emptyset$ if and only if $i = j$. Therefore, the sets $C'_1 \cup C''_1, \ldots, C'_k \cup C''_k$ partition C, and for any kernel V'' of D with $V'' \subset V'$ or $V' \subset V''$ or $V' \cap V'' = \emptyset$ or $V' \cup V'' = V$ there is an arrow in $C'_i \cup C''_i$ entering V'', for each i. To prove that this is true for *each* kernel of D, let V'' be a kernel with

$$d^-_{C'_i \cup C''_i}(V'') = 0 \tag{4.1}$$

for a certain i. So $V' \cap V'' \neq \emptyset$ and $V' \cup V'' \neq V$, and hence $V' \cap V''$ and $V' \cup V''$ are kernels of D again. Also

$$d^-_{C'_i \cup C''_i}(V' \cap V'') + d^-_{C'_i \cup C''_i}(V' \cup V'') \leqslant d^-_{C'_i \cup C''_i}(V') + d^-_{C'_i \cup C''_i}(V'') \,. \tag{4.2}$$

Since $d^-_{C'_i \cup C''_i}(V') = 1$, at least one of the two left terms is 0. But $V' \cap V'' \subset V' \subset V' \cup V''$, and hence both left terms are nonzero.

(ii) *If $a = (v, w)$ belongs to C, then v is a source of D or w is a sink of D.* For suppose not. Then, by (i), a is not in any directed cut intersecting C in exactly k arrows. So removing a from C, by induction, $C \backslash \{a\}$ can be split into k coverings for the directed cuts. Hence also C can be split in such a way.

(iii) *If $a = (v, w)$ and $a' = (v', w')$ belong to C, and (v', w) belongs to A, then v' is a source or w is a sink of D.* For suppose v' is not a source and w is not a sink. By (ii) this implies that v is a source and w' is a sink, and hence $a'' = (v, w')$ belongs to A. Since $a \neq a'$ (as v is a source and v' not), the set $C' = (C\setminus\{a, a'\}) \cup \{a''\}$ is smaller than C. Moreover, $d^-_{C'}(V') \geq k$ for each kernel V' of D, as the number of arrows in C' meeting any source or sink is the same as that for C, and, in general,

$$d^-_{\{a''\}}(V') \leq d^-_{\{a,a'\}}(V') \leq d^-_{\{a''\}}(V') + 1\,. \tag{4.3}$$

So, by induction, C' can be split into classes $C'_1, \ldots, C'_k$ such that $d^-_{C'_i}(V') \geq 1$ for each kernel V' and each i. Assuming $a'' \in C'_1$, we can replace C'_1 by $(C'_1\setminus\{a''\}) \cup \{a, a'\}$, and this yields, by (4.3), a splitting of C as required.

(iv) *There exists a kernel V' for D, containing all sinks but no sources, such that if $(v, w) \in C$ enters V', then v is a source and w is a sink.* For let V' consist of all sinks, together with all vertices u for which there is an arrow (v, w) in C with v not a source, and $(v, u) \in A$. One easily checks that V' is a kernel containing all sinks but no sources. Moreover, suppose $(t, u) \in C$ enters V'. If u is a sink and t is no source, then $t \in V'$, contradicting that (t, u) enters V'. If t is a source and u is not a sink, then, by definition of V', there is an arrow (v, w) in C with v not a source, and $(v, u) \in A$. But this contradicts (iii). Hence t is a source and u is a sink.

(v) Let V' be as in (iv), and let $V'' = V\setminus V'$. Let $D' = (V, A')$ be the directed graph arising from D by replacing any arrow (v, w) of D by k parallel arrows from w to v. One easily checks that

$$d^-_{A'\cup C}(W) \geq k \tag{4.4}$$

for each nonempty proper subset W of V. So, by Theorem 3, the set $A' \cup C$ can be split into k bi-branchings with respect to the splitting V', V''. Let $C_1, \ldots, C_k$ be the intersections of these bi-branchings with C. Hence

$$d^-_{A'\cup C_j}(W) \geq 1 \tag{4.5}$$

for each nonempty proper subset W of V with $W \subset V'$ or $V' \subset W$, and $j = 1, \ldots, k$. We show that each C_j intersects each directed cut, which finishes our proof.

Let W be a kernel for D, and let $j = 1, \ldots, k$. We prove that at least one arrow in C_j enters W. Note that if W contains any source, it contains all sinks.

First suppose that W contains no sources of D. By (4.5),

$$d^-_{A'\cup C_j}(W \cap V') \geq 1\,. \tag{4.6}$$

Since $W \cap V'$ again is a kernel of D, we have $d^-_{A'}(W \cap V') = 0$, and hence there is an arrow in C_j entering $W \cap V'$. Since, by (iv), each arrow in C entering V' starts in a source, and since W does not contain any source, this arrow enters W also.

Second, if W contains every sink of D, by symmetric arguments (now considering $W \cup V'$) again at least one arrow of C_j enters W. □

Remark. There is another special case in which the conjecture of Edmonds and Giles is true, namely if D arises from a directed tree T, with vertex set V, by taking the transitive closure (i.e., $A = \{(v, w) \mid$ there is a directed path in T from v to $w\}$. This can be shown using the total unimodularity of matrices involved. One may ask for a common generalization of this special case and Theorem 4 above.

5. An extension of Theorem 4

We now extend Theorem 4, thus obtaining a common generalization of the Theorems 3 and 4 (cf. Section 0), by the following observation.

Observation. Let $D = (V, A)$ and $D' = (V, A')$ be directed graphs. Let $a = (v, w)$ be an arrow of D such that there exist vertices v' and w', and directed paths in D' from v to v', from w' to v', and from w' to w (cf. Fig. 3, where wriggled lines represent directed paths in D'). The vertices v, v', w', w need not to be distinct.

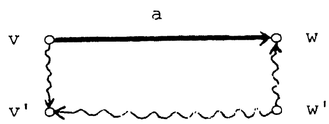

Fig. 3.

Now let v'' and w'' be two new vertices, let $V_0 = V \cup \{v'', w''\}$, $a'' = (v'', w'')$, $A_0 = (A \backslash \{a\}) \cup \{a''\}$, and $A'_0 = A' \cup \{(v, v''), (v'', v'), (w'', v''), (w', w''), (w'', w)\}$, $D_0 = (V_0, A_0)$, $D'_0 = (V_0, A'_0)$ (cf. Fig. 4, where heavy and light lines stand for arrows of D_0 and D'_0, respectively). Then one easily checks that, for each subset A'' of A, A'' is a strong connector for D', if and only if A''_0 is a strong connector for D'_0, where $A''_0 = A''$ if $a \notin A''$, and $A''_0 = (A'' \backslash \{a\}) \cup \{a''\}$ if $a \in A''$. Hence the hypergraphs of strong connectors for the two cases are isomorphic. Therefore, also the hypergraphs of minimal strong cuts are isomorphic (as these are the 'blockers' of the first ones).

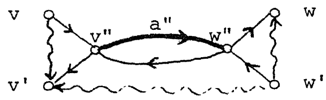

Fig. 4.

This gives us the invariance of certain min–max relations under these transformations. We shall apply this observation to derive the following theorem from Theorem 4, and Theorem 6 from the Lucchesi–Younger theorem.

Theorem 5. *Let $D' = (V, A')$ be an acyclic directed graph, such that each pair of source and sink is connected by a directed path. Let $D = (V, A)$ be a directed graph. Then the maximum number of pairwise disjoint strong connectors for D' is equal to the minimum size of a strong cut induced by D'.*

Proof. We may suppose that D' is transitive, i.e., that if (u, v) and (v, w) are in A', then also (u, w) is in A'. We prove the theorem by induction on the number of arrows $a = (v, w)$ in A with (w, v) not in A'. If this number is 0, the theorem is equivalent to Theorem 4.

So suppose $a = (v, w) \in A$ and $(w, v) \notin A'$. Let v' be a sink of D' with $(v, v') \in A'$, and let w' be a source of D' with $(w', w) \in A'$. By assumption $(w', v') \in A'$, and hence we may make digraphs D_0 and D_0' as in the Observation above. Since strong connectors, and strong cuts, determine isomorphic hypergraphs in the two cases, the conditions of the theorem hold also for D_0 and D_0'. Since the number of arrows in D_0 which do not occur in reversed direction in D_0', is one less than for D and D', we can split A_0 as required, and hence, since the hypergraphs are isomorphic, we can split A as required. □

One easily derives the following weighted version.

Corollary 5a. *Let $D' = (V, A')$ be an acyclic directed graph, such that each pair of source and sink is connected by a directed path. Let $D = (V, A)$ be a directed graph, and let $c : A \to \mathbb{Z}_+$ be a capacity function. Suppose that the minimum capacity of a strong cut induced by D' is at least k. Then there are k strong connectors for D' such that no arrow a is in more than $c(a)$ of these strong connectors.*

Proof. Replace each arrow a of D by $c(a)$ parallel arrows, and apply Theorem 5. □

6. A similar extension of the Lucchesi–Younger theorem

We can apply the Observation of Section 5 also to obtain a somewhat more general form of the Lucchesi–Younger theorem [18] (cf. [16]). The Lucchesi–Younger theorem says that the minimum size of a set of arrows in a directed graph $D=(V,A)$ intersecting each directed cut, is equal to the maximum number of pairwise disjoint directed cuts. It is easy to derive, by replacing arrows by directed paths, from this a weighted version: given a length function $l: A \to \mathbb{Z}_+$, the minimum length of a set of arrows intersecting all directed cuts, is equal to the maximum number of directed cuts such that no arrow a is in more than $l(a)$ of these directed cuts.

The more general theorem is as follows.

Theorem 6. *Let $D=(V,A)$ and $D'=(V,A')$ be directed graphs, such that for each arrow $a=(v,w)$ of D there are vertices v' and w' and directed paths in D' from v to v', from w' to v', and from w' to w. Let $l: A\to\mathbb{Z}_+$ be a length function. Then the minimum length of a strong connector for D' is equal to the maximum number of strong cuts induced by D' such that no arrow a is in more than $l(a)$ of these strong cuts.*

Proof. The proof is similar to that of Theorem 5. □

A direct corollary is another 'bi-branching theorem'.

Corollary 6a. *Let $D=(V,A)$ be a directed graph, and let V be split into classes V' and V''. Let $l: A\to\mathbb{Z}_+$ be a length function such that each bi-branching has length at least k. Then there are nonempty proper subsets $V_1,\ldots,V_k$ of V such that $V_i\subset V'$ or $V'\subset V_i$ for each i, and no arrow a enters more than $l(a)$ of the V_i.*

Proof. Apply Theorem 6, with $A'=\{(v',v'')\mid v'\in V', v''\in V''\}$. □

Direct consequences to Corollary 6a are Fulkerson's branching theorem [9] and König's theorem [15] on minimal coverings in bipartite graphs. Note that, conversely, the cardinality version of Corollary 6a (i.e., $l\equiv 1$) can be derived easily from König's theorem.

7. Sub- and supermodular functions on directed graphs

Edmonds and Giles [3] gave a common generalization of the Lucchesi–Younger theorem [18] (cf. Section 6) and Edmonds' matroid intersection theorem [1], by considering submodular functions defined on the vertex set of a directed graph. In fact, also the extension of the Lucchesi–Younger theorem given above (Theorem 6) may be included in such a framework—see Theorem 7 below.

Note that a collection $\mathcal{F}$ of subsets of a set V, containing $\emptyset$ and V, is closed under unions and intersections, if and only if there is a directed graph $D' = (V, A')$ such that $\mathcal{F} = \{V' \subset V \mid d^-_{A'}(V') = 0\}$. The following theorem extends Theorem 6 above and another theorem of Frank [5].

Theorem 7. *Let $\mathcal{F}$ be a collection of subsets of V and let f be an integral function defined on $\mathcal{F}$, such that if $V_1, V_2 \in \mathcal{F}$ and $V_1 \cap V_2 \neq \emptyset$, $V_1 \cup V_2 \neq V$, then $V_1 \cap V_2 \in \mathcal{F}$, $V_1 \cup V_2 \in \mathcal{F}$ and $f(V_1 \cap V_2) + f(V_1 \cup V_2) \geq f(V_1) + f(V_2)$. Let furthermore a directed graph $D = (V, A)$ be given such that if $V_1, V_2, V_3 \in \mathcal{F}$ with $V_1 \cap V_2 \cap V_3 = \emptyset$ and $V_1 \cup V_2 \cup V_3 = V$, then no arrow of D enters both V_1 and V_2. Let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a set $A'' \subset A$ such that each $V' \in \mathcal{F} \setminus \{\emptyset, V\}$ is entered by at least $f(V')$ arrows in A'', is equal to the maximum value of*

$$\sum_{i=1}^{k} f(V_i), \tag{7.1}$$

where $V_1, \ldots, V_k$ are sets in $\mathcal{F} \setminus \{\emptyset, V\}$ such that each arrow a of D enters at most $l(a)$ of the V_i.

(The theorem asserts that both sides of a certain linear programming duality equation are achieved by integral solutions—cf. Section 8.)

Theorem 7 can be proved with the standard methods (using cross-free collections, tree-representations, total dual integrality), as described by Edmonds and Giles [3].

Note that the condition given in the second sentence of Theorem 7 is just the analogue of the condition given in the first sentence of Theorem 6. In order to obtain a similar generalization of Theorem 5, one easily checks that a collection $\mathcal{F}$, closed under unions and intersections, is the collection of sets V' with $d^-_{A'}(V') = 0$ for some digraph $D' = (V, A')$ with the property that, after contracting the strong components of D', each pair of source and sink is connected by a directed path, if and only if there are no sets V_1, V_2, V_3 in $\mathcal{F} \setminus \{\emptyset, V\}$ with $V_1 \cap V_2 \cap V_3 = \emptyset$ and $V_1 \cup V_2 \cup V_3 = V$.

Now the following possible generalization of Theorem 5 is *not true*: let $\mathcal{F}$ be a collection of subsets of V with the properties described in the previous paragraph, let f be a supermodular function on $\mathcal{F}$, and let $D = (V, A)$ be a directed graph, such that each set V' in $\mathcal{F} \setminus \{\emptyset, V\}$ is entered by at least $f(V')$

arrows of D. Suppose $f=f_1+f_2$, where f_1 and f_2 are nonnegative integral supermodular functions on $\mathscr{F}$. Then A can be split into classes A_1 and A_2 such that each set V' in $\mathscr{F}\backslash\{\emptyset, V\}$ is entered by at least $f_i(V')$ arrows in A_i, for $i=1,2$. A counterexample to this is given by taking D as in Fig. 5, $\mathscr{F}$ being the collection of all subsets of $V\backslash\{r\}$, and $f=f_1+f_2$, where, for V' in $\mathscr{F}$, $f_1(V')=1$, $f_2(V')=1$ if $s\in V'$, and $f_2(V')=0$ if $s\notin V'$.

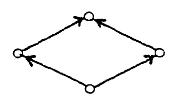

Fig. 5.

The following generalization of Theorem 2 and Theorem 5 might be true.

> Let $\mathscr{F}$ be a collection of subsets of a set V, closed under unions and intersections, such that for no V_1, V_2, V_3 in $\mathscr{F}\backslash\{\emptyset, V\}$ both $V_1\cap V_2\cap V_3=\emptyset$ and $V_1\cup V_2\cup V_3=V$. Let f be a submodular function on $\mathscr{F}$ such that $f(V')\geq k$ for each V' in $\mathscr{F}\backslash\{\emptyset, V\}$. Let $D=(V,A)$ be a directed graph such that $d_A^-(V')\leq f(V')$ for each V' in $\mathscr{F}\backslash\{\emptyset, V'\}$. Then A can be split into classes $A_1,\ldots,A_k$ such that for each V' in $\mathscr{F}\backslash\{\emptyset, V\}$ one has $\Sigma_{j=1}^{k}\max\{d_{A_j}^-(V'),1\}\leq f(V')$. (7.2)

By taking $f(V')=d_A^-(V')$ Theorem 5 follows. By taking A to be a collection of disjoint arrows, with set V' of heads, and $\mathscr{F}$ to be the collection of all V'' with $V''\subset V'$ or $V'\subset V''$, Theorem 2 follows.

The question remains whether both the generalizations of Edmonds–Giles type, and assertions of the type of Theorem 2 and problem (7.2) above, fit into one framework. Also at another point submodular functions, or rather matroids, appear, namely at Fulkerson's branching theorem. This theorem may be interpreted as a min–max relation for the minimum weight of a common base of two matroids (cf. [1]). One may ask whether the more general bi-branching theorem (Corollary 6a), or even Theorem 6, can be formulated in such a way.

8. Polyhedral representations and polynomial algorithms

As usual with min–max relations, Theorems 5 and 6 above allow a polyhedral formulation, or, equivalently, a formulation in terms of linear programming. By the ellipsoid method as described in [10] this often yields the existence of polynomial algorithms.

Let D and D' be as in Theorem 6, and let $c : A \to \mathbb{Z}_+$. Consider the linear programming problem of finding

$$\min \sum_{a \in A} c(a)x(a) \tag{8.1}$$

where $x : A \to \mathbb{Q}_+$ such that

$$\begin{aligned} &0 \leqslant x(a) \leqslant 1 \quad \text{if } a \in A, \\ &\sum_{a \in A''} x(a) \geqslant 1 \quad \text{if } A'' \text{ is a strong cut}, \end{aligned} \tag{8.2}$$

where we mean a 'strong cut' to be induced by D'. By the Duality Theorem of linear programming, (8.1) is equal to

$$\max \sum_{A'' \text{ strong cut}} y(A''), \tag{8.3}$$

where, for each strong cut A'', $y(A'')$ is a rational number such that

$$\begin{aligned} &y(A'') \geqslant 0, \quad \text{if } A'' \text{ is a strong cut}, \\ &\sum_{A'' \ni a} y(A'') \leqslant c(a), \quad \text{if } a \in A. \end{aligned} \tag{8.4}$$

Now, Theorem 6 asserts that (8.1) and (8.3) are attained by *integral* functions x and y. So the system of linear inequalities (8.2) is totally dual integral (cf. [3]), and a function x satisfies (8.2) if and only if x is a convex linear combination of incidence vectors of strong connectors for D'.

If, moreover, D' is acyclic and each pair of source and sink of D' is connected by a directed path, we obtain similar conclusions if we exchange the terms 'strong cut' and 'strong connector', as follows from Corollary 5a. Note that in the latter case, by the theory of blocking polyhedra of Fulkerson [8], if D and D' satisfy the weaker conditions of Theorem 6 only, (8.1) is attained by an integral vector x (i.e., by the incidence vector of some strong cut).

Therefore, by the ellipsoid method there exists a polynomial algorithm for finding minimum length strong connectors, if and only if there exists a polynomial algorithm for finding minimum capacitated strong cuts. However, the existence of the latter algorithm follows easily from the Ford–Fulkerson min-cut algorithm (by giving the arrows of D' sufficiently large capacity), and hence minimum length strong connectors can be found in polynomial time. Also a maximum packing of strong cuts (i.e., an integer solution for (8.3)) can be found in polynomial time, by applying the usual techniques of making cuts cross-free (cf. [10]). Clearly, minimum length strong connectors and maximum packings of strong cuts can be found also by adapting (e.g., by the Observation of Section 5) the existing polynomial algorithms for the Lucchesi–Younger theorem [6, 12, 17].

It remains to show that the splitting of A as described in Theorem 5 and Corollary 5a can be found efficiently. However, our proof above yields a polynomial algorithm. Indeed, the proof of Theorem 5 reduces this theorem to Theorem 3. Since this reduction can be carried out in polynomial time, we need to show that a splitting into bi-branchings can be found efficiently. But the splitting into bi-branchings is obtained by first splitting the 'crossing arrows' (from V_2 to V_1), which splitting can be found by Theorem 2. After that this splitting is extended to a splitting into bi-branchings by Theorem 1. Now to derive polynomial algorithms from the proofs of Theorem 1 and Theorem 2, one needs only a method to find one, or all, minimal nonempty subsets V' with $f(V')=h(V')$, where f is submodular and h is supermodular, with $h \leqslant f$. But this can be reduced easily to the problem of finding a set minimizing a submodular set-function, and this can be solved in polynomial time [10].

Also the splitting described in Corollary 5a, i.e., an integral solution y for (8.3), with strong connectors instead of strong cuts, can be found in time polynomially bounded by the size of the problem. Note that this size is

$$|V|+|A|+|A'|+\sum_{a\in A}\log(c(a)+1)\,, \tag{8.5}$$

so, to obtain a good algorithm, we cannot just replace each arrow a by $c(a)$ parallel arrows. However, by the ellipsoid method a fractional solution y of (8.3) (again with 'strong connector' instead of 'strong cut'), can be obtained in polynomial time, such that the number of strong connectors A'' with $y(A'')>0$ is at most $|A|$. Now let

$$c'(a):=\sum_{A''\ni a}(y(A'')-\lfloor y(A'')\rfloor)\,, \tag{8.6}$$

where the sum ranges over strong connectors A'', and where $\lfloor\ \rfloor$ denotes lower integer part. Since $c'(a)\leqslant|A|$ we can replace each arrow a by $c'(a)$ parallel arrows, and then find in this new directed graph as many as possible pairwise disjoint strong connectors, by the method described above for Theorem 5, i.e., we find integers $y'(A'')\geqslant 0$ for each strong connector A''. One easily checks that $\lfloor y(A'')\rfloor + y'(A'')$ is an integer solution for (8.3).

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